AP Calculus AB
Lessons 3-1, 3-2, 5-3 *Relating the Graphs of* f, f', and f'' *Part 2*

Name Hem 1 2016
Date

Learning Goals:

- *I can use the first and second derivative tests to determine local extreme values of a function.*
- *I can determine the concavity of a function and locate points of inflection by analyzing the second derivative.*

***Directions*: Below are passages from a textbook (not our text book) about relating the graphs f, f', and f'', along with questions and practice problems. Work through the following problems in your group, asking for help when necessary!**

* *Note: The ◄ in the text are placed where it is suggested that you "check this fact yourself" – meaning it is a good place to stop and assess if you understand what you are reading!!*

The Geometry of Derivatives

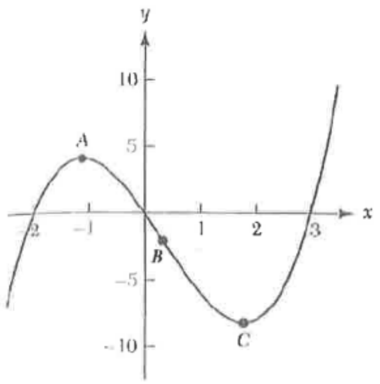
The geometric relationship between a function f and its derivative f' is easy to state:

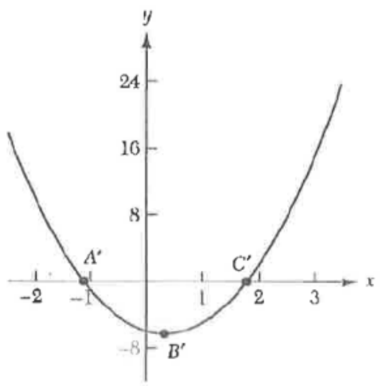
> *For any input a, $f'(a)$ is the slope of the line tangent to the f-graph at $x = a$.*

This statement—innocuous as it appears—is one of the most important in this book. The next two sections (and much of this book) explore its meaning and implications.

Graphs of f and f': An Extended Example

Graphs of a function f and its derivative f' follow. (For the moment, no formulas are given—or needed. We'll return to these functions symbolically at the end of the next section.) ◄ Three interesting points are labeled on each. ◄





Based on our prior learning, why are the points labeled on each graph "interesting"? How are the points A and C related to A' and C'? We will learn about the relationship between B and B' in this packet . . .

A&C are relative extrema of f, which are the zeros of f'. These points are A' & C'.

OVER →

The f'-graph tells how slopes of tangent lines to the f-graph behave. At B, for instance, the f-graph seems ► to have a slope around -6; for this reason, the f'-graph has height -6 at B'.

First let's observe several straightforward geometric relationships between the two graphs, introducing some useful new terminology in the process.

Why does the graph of f' have a height of -6 when the graph of f has a tangent line with a slope of -6?

f' is the slope of the T.L.'s to f - so all the y-values for f' are slopes of f.

The Sign of f' The graph of f rises to the left of A, falls between A and C, and rises again to the right of C. The sign of $f'(x)$ tells whether the line tangent to the f-graph at x points up or down. At A' and C', ► f' changes sign. Thus, at A and C, f itself changes direction.

Why does f change directions when f' changes sign? Explain in terms of the meaning of a derivative.

If the slope of the T.L. is positive, f is increasing; if the slope of the T.L. is negative, f is decreasing.

Stationary, Maximum, and Minimum Points The points A and C, with approximate coordinates $(-1.1, 4)$ and $(1.8, -8)$, ► where the f-graph is horizontal, are obviously of interest. They mark, respectively, high and low points of the graph. The situation looks clear, but to avoid later confusion, it will pay to be extremely picky with language here—especially about the distinction between inputs to f (called points) and outputs from f (called values).

Here, in full detail, is the situation at $A \approx (-1.1, 4)$. The domain point $x = -1.1$ is called a local maximum point of f; the corresponding output—$f(-1.1) \approx 4$—is called a local maximum value of f. At C ► the situation is similar: $x = 1.8$ is a **local minimum point** of f, and $f(1.8) \approx -8$ is the corresponding **local minimum value** of f. The x-coordinates of both A and C are called **stationary points** of f. ◄ (We say local rather than global because elsewhere in its domain f may assume larger or smaller values.) The corresponding points A' and C' occur where the f'-graph crosses the x-axis (i.e., at roots of f').

*What is the difference between a **local minimum point** and a **local minimum value**? Why do you think the author makes a point of being "picky" with the vocabulary?*

point ⇒ x-value

value ⇒ y-coordinate

The value of the function means y-coordinate.

Concavity and Inflection The point B, near $x = 0.3$, is an **inflection point** of f: At B, the f-graph's **direction of concavity** changes, from concave down to concave up. ◄ The point B has another special geometric property: At B, the graph of f points most steeply downward.

The corresponding point on the f'-graph, B', is easier to see; it's a local minimum point. Later we'll use this property and some algebra to find the *exact* location of B. ◄

One informal way to explain concavity is to think of concave up as where the graph "holds water" and concave down as where the graph "spills water". Give a definition of concave up and concave down in your own words.

Concave up ⇒ T.L. under the graph

Concave down ⇒ T.L. above the graph.

What f' Says about f

Interpreting the derivative function f' in terms of the slopes of tangent lines on the f-graph has many important geometric implications. We summarize several below.

Increasing or Decreasing?

A function f **increases** where its graph rises ◄ and **decreases** where its graph falls. The following definition captures these natural ideas in analytic language.

Definition: Let I denote the interval (a,b).

A function is **increasing** on I if $f(x_1) < f(x_2)$ whenever $a < x_1 < x_2 < b$

A function is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $a < x_1 < x_2 < b$

Explain what $a < x_1 < x_2 < b$ means.

x_1 is less than x_2; both are between a & b

Explain what $f(x_1) < f(x_2)$ and $f(x_1) > f(x_2)$ means.

y of f at x_1 is less than y of f at x_2.

y of f at x_1 is greater than y of f at x_2.

In your own words, what does it mean for a graph to be increasing? For a graph to be decreasing?

Increasing ⇒ slope of T.L. is positive

Decreasing ⇒ slope of T.L. is negative

OVER →

Fact If $f'(x) > 0$ for all x in I, then f increases on I. If $f'(x) < 0$ for all x in I, then f decreases on I.

This fact certainly *sounds* reasonable. To say that $f'(x) > 0$ means that the tangent line to the f-graph at x points upward. With any luck, so should the f-graph itself.

Behavior at a Point. Functions are often said to increase or decrease *at a point*. To say, for example, that f increases *at* $x = 3$ means that f increases on some interval—perhaps a small one—containing $x = 3$. Now we can restate the preceding fact ▶ as follows:

Fact If $f'(a) > 0$, then f is increasing at $x = a$. If $f'(a) < 0$, then f is decreasing at $x = a$.

At $x = 1$, for instance, $f'(1) < 0$. The Fact says—and the picture agrees—that at $x = 1$ the f-graph is decreasing.

The converse of the above fact is as follows

Fact If f increases at $x = a$, then $f'(a) \geq 0$; if f decreases at $x = a$, then $f'(a) \leq 0$.

Explain in your own words what the previous two "Facts" tell us about the relationship between the graph of a function and the graph of the functions derivative.

If f is increasing, f′ is positive.
If f is decreasing, f′ is negative.
(And vice-versa)

Finding Maximum and Minimum Points

Geometric intuition says that at a local maximum point or local minimum point, a smooth graph must be "flat." ▶ More succinctly:

Fact On a smooth graph every local maximum or local minimum point x_0 is a stationary point—i.e., a root of f'.

This fact has immense practical value. To find maximum and minimum values of f, the fact says, we can limit our search to roots of f'. Each such root is a stationary point and therefore, possibly, a maximum or minimum point (but not a sure thing—a stationary point may be only a "flat spot" in the graph). ▶

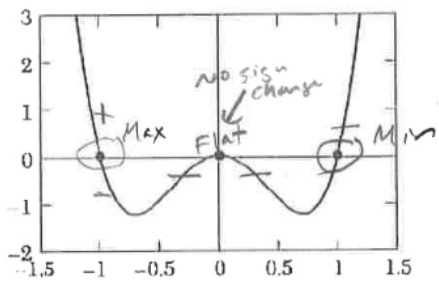
The next example—an important one—shows how to sort out all the possibilities.

Explain what the above "Fact" means in terms of finding local maximum/minimum points of graphs using the calculus we've learned – how can we find local maximum and minimum points of any function f? How can we tell if the point is a maximum or minimum?

Set the derivative equal to zero and solve for x.

EXAMPLE 2 The graph of a function f' appears as follows; the f-graph is not shown (for now). Three points of interest are bulleted. Where, if anywhere, does f have local maximum or local minimum points? Why?

Graph of f'



Solution: The three bullets on the graph—at $x = -1$, $x = 0$, and $x = 1$—represent roots of f' and therefore correspond to stationary points of f. What type of stationary point is each one: a local maximum, a local minimum, or just a flat spot? The key to deciding is to check the sign of f' just before and just after each stationary point. We take each root of f' in turn.

At $x = -1$ Just before (i.e., to the left of) $x = -1$, $f'(x) > 0$. Therefore (by an earlier Fact), f increases until $x = -1$. Just after $x = -1$, $f'(x) < 0$, so f decreases immediately after $x = -1$. This means that f has a local maximum at $x = -1$.

At $x = 1$ Consider values of x near $x = 1$. The graph shows that if $x < 1$, $f'(x) < 0$; if $x > 1$, $f'(x) > 0$. Thus, reasoning as above, f decreases before $x = 1$ and increases after $x = 1$. This means that f has a local minimum at $x = 1$.

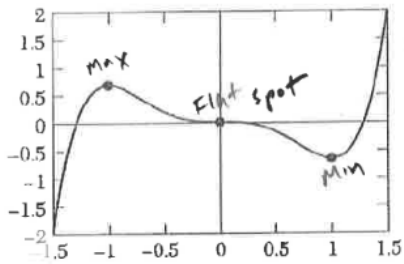
At $x = 0$ This time the graph shows that $f'(x) < 0$ on both sides of $x = 0$. Thus, f must decrease before *and* after $x = 0$, so $x = 0$ is neither a maximum nor a minimum point, but just a flat spot in the f-graph.

Note above the explanation for why f does NOT have a local max/min at x = 0!! This is more in depth than we discussed in the previous investigation. Be sure to understand what is happening to the graph of f at x = 0.

OVER →

Here, at last, is a possible f-graph. It agrees with everything we said.

Graph of f



Let's summarize what we know about stationary points.

Fact (First Derivative Test) Suppose that $f'(x_0) = 0$.

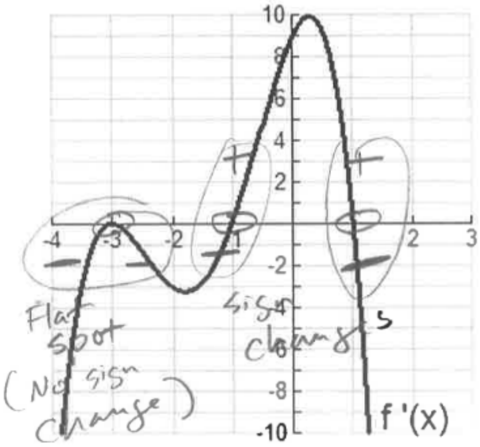
- If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, then x_0 is a local minimum point.
- If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then x_0 is a local maximum point.

Practice: Given the below graph of $f'(x)$, find all the local maximum points, minimum points, and "flat spots" of $f(x)$

Flat spot at $x = -3$ (dec/dec)

Local minimum at $x = -1$ (dec/inc) ∪

Local maximum at $x = 1$ (inc/dec) ∩



Check your answer with The Heinl before you move on to the next page!!

Concave Up or Concave Down?

So far we've described concavity and inflection points informally, in graphical language. Here's a more formal, analytic definition:

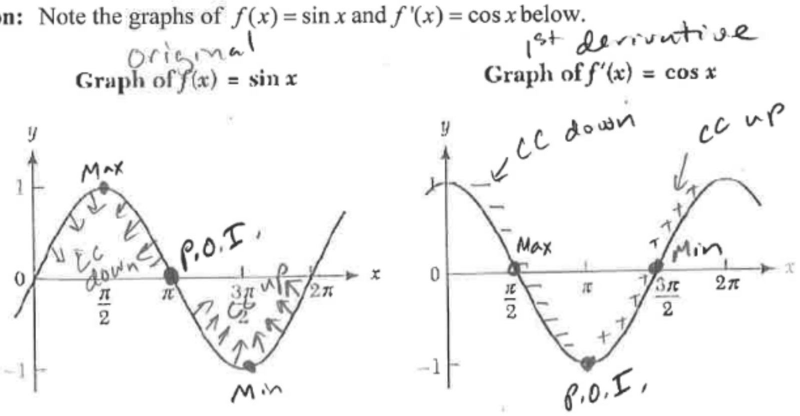
Definition: The graph of f is **concave up** at $x = a$ if the derivative function f' is increasing at $x = a$.
The graph of f is **concave down** at $x = a$ if f' is decreasing at $x = a$.
Any point at which a graph's direction of concavity changes is called an **inflection point.**

Finding Inflection Points from the Graph of f'

The direction of concavity of the graph of f depends, as the definitions show, on whether f' increases or decreases. An inflection point occurs wherever *f' changes direction*—i.e., wherever f' has a local minimum or a local maximum.

Example: We know that the derivative of the sine function is the cosine function. That is, if $f(x) = \sin x$, then $f'(x) = \cos x$. Based on this knowledge, discuss the concavity of the sine function. Find all inflection points and describe them in derivative language.

Solution: Note the graphs of $f(x) = \sin x$ and $f'(x) = \cos x$ below.



Notice:

Stationary Points f has stationary points (a local ▶ maximum point and a local minimum point) $x = \pi/2$ and at $x = 3\pi/2$—exactly the roots of f'.

Increasing or Decreasing? f increases on the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$; on the same intervals, f' is positive.

Concavity f is concave down on $(0, \pi)$—where f' decreases—and concave up on $(\pi, 2\pi)$—where f' increases. In fact, f illustrates every possible combination of increasing/decreasing and concavity behavior.

Inflection Points f has an inflection point at each multiple of π—precisely where f' assumes a local maximum or local minimum. ■

Annotate (make notes on) the graphs above that illustrate the stationary points, increasing and decreasing intervals, concavity, and inflection points.

OVER →

If f' has a local maximum or minimum, what is the value of f''? $f'' = 0$

Based on your answer to the above question, how can you find inflection points using the second derivative?

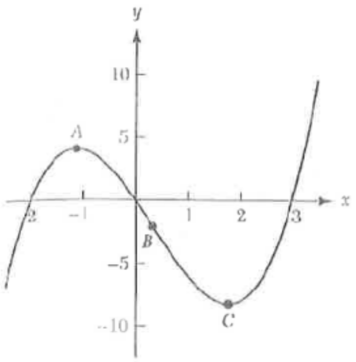
Set the 2nd derivative equal to zero & solve for x.

What do your inflection points tell you about the graph of f?

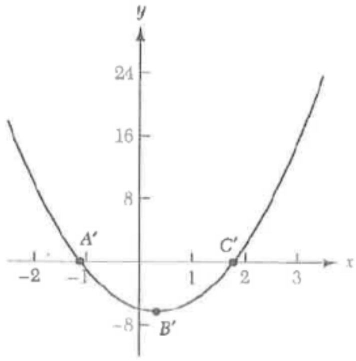
Where f changes from concave up to concave down.

Practice: The graphs of f and f' are reprinted below. They again illustrate the definition of concavity given above.

Graph of f



Graph of f'



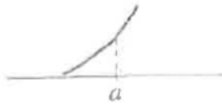
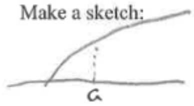
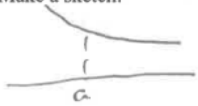
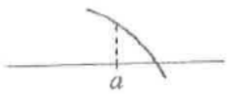
Explain, referencing the labeled points on the above graphs, how the graph of f' illustrate the stationary points, increasing and decreasing intervals, concavity, and inflection points of the graph of f.

f' is decreasing from (-∞, B')
f is concave down from (-∞, B)

f' is increasing from (B', ∞)
f is concave up from (B, ∞)

f' has a min at B'
f changes concavity at B

Fill in the below table based on what you have learned about the first and second derivative:

	Conditions on the Derivatives	Description of $f(x)$ at $x=a$	Graph of $y=f(x)$ near $x=a$
1.	$f'(a)$ is positive $f''(a)$ is positive	$f(x)$ is increasing $f(x)$ is concave up	
2.	$f'(a)$ is positive $f''(a)$ is negative	$f(x)$ is increasing $f(x)$ is concave down	Make a sketch: 
3.	$f'(a)$ is negative $f''(a)$ is positive	$f(x)$ is decreasing $f(x)$ is concave up	Make a sketch: 
4.	$f'(a)$ is negative $f''(a)$ is negative	$f(x)$ is decreasing $f(x)$ is concave down	

Check your answer with The Heinl before you move on!!

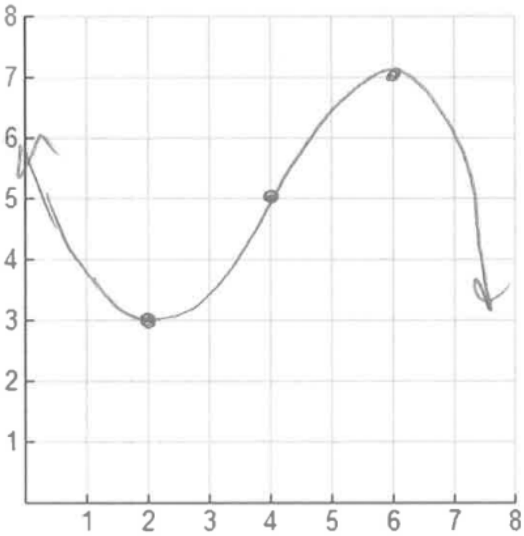
Practice: Sketch a graph of a function $f(x)$ with all the following properties:

a. $(2,3)$, $(4,5)$, and $(6,7)$ are on the graph.

b. $f'(6)=0$ and $f'(2)=0$

c. $f''(x)>0$ for $x<4$, $f''(4)=0$, and $f''(x)<0$ for $x>4$.

concave down ∩   P.O.I.   concave up ∪



OVER →

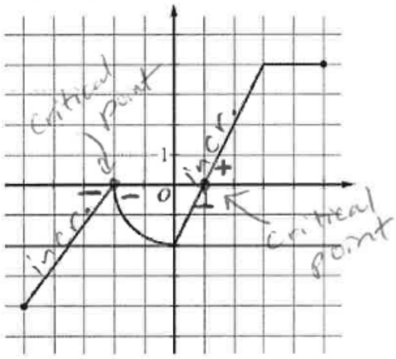
Previously we described the "First Derivative Test". Use it as a model to write the "Second Derivative Test". Check your answer with The Heinl before moving on.

Second Derivative Test: Suppose $f''(x) = 0$
- If $f''(x) < 0$ for $x < x_0$ and $f''(x) > 0$ for $x > x_0$, then x_0 is an inflection point when f changes from concave down to concave up.
- Same (with inequality symbols switched) for concave up.

2017 Exam: FRQ #3 (No calculator)

Graph of g'

☆ Derivative graph



- every single point is this graph represents the slope of all of the tangent lines on g.

The graph of g', consisting of three line segments and a quarter of a circle, is shown above.

(b) Find the instantaneous rate of change of g with respect to x at $x = 3$, or state that it does not exist.

(c) On what open intervals, if any, is the graph of g concave up? Justify your answer. → when the first deriv. is increasing

(d) Find all x-values in the interval $-5 < x < 5$ at which g has a critical point. Classify each critical point as the location of a local minimum, a local maximum, or neither. Justify your answers. → when $g'(x) = 0$ AKA: x-intercepts of $g'(x)$

→ must state there is a change in signs

(1pt) b. $g'(3) = 4$
The instantaneous rate of change of g at $x = 3$ is 4.

(2pts intervals + justific.) c. g is concave up when g' is increasing. g' is increasing on $-5 < x < -2$ and $0 < x < 3$.
(☆ ok to write $(-5, 2)$ and $(0, 3)$)

(3pts) d. Critical points happen when $g'(x) = 0$.
$g'(x) = 0$ when $x = -2$ and $x = 1$. Therefore g has critical points at $x = -2$ and $x = 1$.
- g has neither a local max nor a local min at $x = -2$ b/c g' does not change signs there.
- g has a local minimum at $x = 1$ because g' changes from negative to positive there.

state $g'(x) = 0$, critical pts, answers w/ just.